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STRESS BASED NONLOCAL INTERACTIONS

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Abstract. The progressive degradation of quasi-brittle materials can be reproduced efficiently by means of damage models. The presence of microcracks in the media gives a nonlocal aspect to the evolution of damage. By interacting with each other under loading, it leads to stress amplification (singularity) or decrease (shield effect) at a given location. In the framework of damage models, the nonlocal integral method [1] or the gradient enhanced media [2] introduce this notion of interactions between points by means of an internal length.

However, they are still some pending issues regarding these methods [3] (e.g., treatment of free boundaries, description of the damage state close to complete failure). In this paper, a modification of the nonlocal integral regularization method is proposed. The influence of a point on its neighbourhood is evolving during the loading and its intensity and direction depend on the stress state it encounters.

Through several numerical simulations, we show that our proposition improves the treatment of free boundaries and gives physically sound damage and strain field in the fracture process zone up to complete failure. The latter being a key parameter of durability analysis.

1 INTRODUCTION

Quasi-brittle materials show the presence of microcracks in their media. Under loading, these microcracks interact with each other, leading to nonlocal interactions. During the cracking, strain localization appears with a size and an orientation of the localized band as well as its evolution that can be directly linked to the nonlocal interactions due to microcracks.

In continuous media, the microcracks are not explicitly represented. As a consequence, additional generalized constitutive equations need to be introduced in the models to take into account the nonlocal character of the propagation and coalescence of microdefects. These models replace the local internal variable by its nonlocal counterpart. For the

nonlocal gradient model [2], the nonlocal internal variable fulfills a differential equation whereas for the nonlocal integral model [1], the nonlocal internal variable is a weighted spatial average.
In addition to restoring the objectivity of the numerical modeling for strain softening behavior, these models aim at describing the behavior of quasi-brittle materials for microcracked area which do not degenerate into a widely opened crack and size effect through the introduction of an internal length.
However, several drawbacks arise from the original models (e.g., description of the kinematic fields in the FPZ, damage initiation in crack tip-problem, description of the interactions in the vicinity of boundaries). To overcome these problems, we propose a new nonlocal integral method in which the weighting is enhanced by introducing the influence of the stress state on the interactions.
First, the original model associated to a damage model is recall. Then, the stress based nonlocal model is presented. Finally, several tests are performed addressing the different drawbacks quoted previsouly.

2 NONLOCAL DAMAGE MODEL

2.1 Continuum damage theory

A scalar isotropic damage model for describing the non linear behavior of concrete under monotonic loading is used. The general stress-strain relationship is:

$$\sigma_{ij} = (1 - D)C_{ijkl} : \varepsilon_{kl} \tag{1}$$

where σ_{ij} and ε_{kl} are the components of the Cauchy stress tensor and the strain tensor, respectively (i, j, k, l $\in [1, 3]$) and C_{ijkl} are the components of the fourth-order elastic stiffness tensor.
The evolution of D is driven by an equivalent strain ε_{eq} that quantifies the local deformation state in the material. Among several definitions, we consider here the equivalent strain defined by Mazars with its corresponding evolution law [6].
The damage scalar variable D is a function of the internal variable Y. This parameter equals the damage threshold ε_{D_0} initially. Its evolution is governed by the Kuhn-Tucker condition:

$$\varepsilon_{eq} - Y \leq 0\ ,\ \dot{Y} \geq 0\ ,\ \dot{Y}(\varepsilon_{eq} - Y) = 0 \tag{2}$$

Mazars has introduced a local measure ε_{eq} of the strain tensor defined by:

$$\varepsilon_{eq} = \sqrt{\sum_{i=1}^{3} \langle \varepsilon_i \rangle_+^2} \tag{3}$$

$\langle \varepsilon_i \rangle_+$ denotes the positive principal strains. This model considers that damage is driven by positive strains, i.e. extension. It allows to accurately reproduce the behavior of quasi-brittle materials such as concrete. In this model, damage is determined as a linear combination of two damage variables (Eq. 4): D_t and D_c which are damage due to tension and compression respectively [6]:

$$D = \alpha_t D_t + \alpha_c D_c \tag{4}$$

The parameters α_t and α_c depend on the stress state (e.g. $\alpha_t = 1$ in pure traction). The damage evolution is characterized by the following exponential law:

$$D_{c,t} = 1 - \frac{\varepsilon_{D_0}(1 - A_{c,t})}{Y} - \frac{A_{c,t}}{\exp^{\left(B_{c,t}(Y - \varepsilon_{D_0})\right)}} \tag{5}$$

A_t, B_t, A_c and B_c are the parameters governing the shape of the evolution law. The constitutive relation exhibits strain softening and as a consequence, needs a regularization technique.

2.2 Original integral nonlocal approach

In the nonlocal damage model, the equivalent strain given in Eq. 3 is replaced by an average equivalent strain $\bar{\varepsilon}_{eq}$ over a volume Ω in the equation governing the growth of damage as defined by Pijaudier-Cabot and Bažant [1].

$$\bar{\varepsilon}_{eq}(\boldsymbol{x}) = \frac{\int_\Omega \phi(\boldsymbol{x} - \boldsymbol{s}) \varepsilon_{eq}(\boldsymbol{s}) d\boldsymbol{s}}{\int_\Omega \phi(\boldsymbol{x} - \boldsymbol{s}) d\boldsymbol{s}} \tag{6}$$

$\phi(\boldsymbol{x} - \boldsymbol{s})$ is the weight function defining the interaction between the considered point located at $\boldsymbol{x}$ and the neighboring points located at $\boldsymbol{s}$. The most used nonlocal weight function is taken as the Gauss distribution function:

$$\phi(\boldsymbol{x} - \boldsymbol{s}) = \exp\left(- \left(\frac{2\, \|\boldsymbol{x} - \boldsymbol{s}\|}{l_c} \right)^2 \right) \tag{7}$$

where l_c is the internal length of the model.

2.3 Stress based nonlocal integral approach

In the proposed approach, the point of view of the calculation of nonlocal quantities is slightly different. *We no longer consider what a point located at $\boldsymbol{x}$ can receive but what a point located at $\boldsymbol{s}$ can distribute.* The nonlocality is defined as a quantity given by each point located at $\boldsymbol{s}$ along its principal stress direction with an intensity depending on the level of the principal stress. We introduce in the nonlocal regularization both the notion of directionality as shown by Pijaudier and Dufour [7] in the limited case of the vicinity of boundaries and the variation of the intensity depending on the state of loading in the structure. The stress field allows the direct description of the presence of free boundary and the development of fracture process zone that are at the origins of

the modification of the nonlocal interactions. During the calculation, the evolution of the interactions between points is considered through a single scalar ρ that, multiplied by the characteristic length l_c, defines the internal length of the model. This internal length evolves from zero for stress-free material up to l_c when maximum principal stress is reached. It is important to notice that this coefficient depending on the stress state of the distributed points does not introduce any parameter in the model.
Let us denote $\boldsymbol{\sigma}_{prin}(\boldsymbol{s})$, the stress state of the point located at $\boldsymbol{s}$, expressed in its principal frame. The vectors forming this frame are $\boldsymbol{u}_1(\boldsymbol{s})$, $\boldsymbol{u}_2(\boldsymbol{s})$, and $\boldsymbol{u}_3(\boldsymbol{s})$ with the associated principal stresses $\sigma_1(\boldsymbol{s})$, $\sigma_2(\boldsymbol{s})$ and $\sigma_3(\boldsymbol{s})$.

$$\boldsymbol{\sigma}_{prin}(\boldsymbol{s}) = \sum_{i=1}^{3} \sigma_i(\boldsymbol{s})(\boldsymbol{u}_i(\boldsymbol{s}) \otimes \boldsymbol{u}_i(\boldsymbol{s})) \tag{8}$$

where $\otimes$ is the tensor product. We define an ellipsoid centered at point $\boldsymbol{s}$, corresponding to a homothety of the original interaction domain with a ratio $|\frac{\sigma_i(\boldsymbol{s})}{f_t}|$ along principal stress direction $\boldsymbol{u}_i(\boldsymbol{s})$. f_t denotes the tensile strength of the material.
The choice of f_t leads to no modification of the interactions at the tensile stress peak, in the direction associated to the maximum tensile stress. The characteristic length l_c associated to the material defines the maximum size of the domain of interactions and so the internal length ρl_c of the stress based nonlocal model can not exceed this value. As a consequence, we need to limit in compression the value of ρ to one under loading directions for which $|\sigma_i(\boldsymbol{s})|$ is higher than f_t.
By using the spherical coordinates (ρ, θ and ϕ), the following equation describes the ellipsoid associated to the stress state of the point located at $\boldsymbol{s}$ (Fig. 1).

$$\rho(\boldsymbol{x}, \boldsymbol{\sigma}_{prin}(\boldsymbol{s}))^2 = \frac{1}{f_t^2\left(\frac{\sin^2\varphi\cos^2\theta}{\sigma_1^2(\boldsymbol{s})} + \frac{\sin^2\varphi\sin^2\theta}{\sigma_2^2(\boldsymbol{s})} + \frac{\cos^2\varphi}{\sigma_3^2(\boldsymbol{s})}\right)} \tag{9}$$

where θ is the angle between $\boldsymbol{u_1}$ and the projection of $(\boldsymbol{x} - \boldsymbol{s})$ onto the plane defined by $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ and φ is the angle between $\boldsymbol{u_3}$ and $(\boldsymbol{x} - \boldsymbol{s})$. Considering, these angles, we obtain:

$$\cos\theta = \frac{\boldsymbol{u}_1.(\boldsymbol{u}_3 \wedge ((\boldsymbol{x}-\boldsymbol{s}) \wedge \boldsymbol{u}_3))}{\|\boldsymbol{u}_3 \wedge ((\boldsymbol{x}-\boldsymbol{s}) \wedge \boldsymbol{u}_3)\|} \qquad \sin\theta = \frac{\boldsymbol{u}_2.(\boldsymbol{u}_3 \wedge ((\boldsymbol{x}-\boldsymbol{s}) \wedge \boldsymbol{u}_3))}{\|\boldsymbol{u}_3 \wedge ((\boldsymbol{x}-\boldsymbol{s}) \wedge \boldsymbol{u}_3)\|} \tag{10}$$

$$\cos\varphi = \frac{\boldsymbol{u}_3.(\boldsymbol{x}-\boldsymbol{s})}{\|\boldsymbol{x}-\boldsymbol{s}\|} \qquad \sin\varphi = \frac{(\boldsymbol{x}-\boldsymbol{s}).(\boldsymbol{u}_3 \wedge ((\boldsymbol{x}-\boldsymbol{s}) \wedge \boldsymbol{u}_3))}{\|(\boldsymbol{x}-\boldsymbol{s})\| \,.\, \|\boldsymbol{u}_3 \wedge ((\boldsymbol{x}-\boldsymbol{s}) \wedge \boldsymbol{u}_3)\|}$$

where $\wedge$ is the vector product and "." is the scalar product. The weight function now reads:

$$\phi(\boldsymbol{x}-\boldsymbol{s}) = \exp\left(-\left(\frac{2\,\|\boldsymbol{x}-\boldsymbol{s}\|}{l_c\;\rho(\boldsymbol{x}, \boldsymbol{\sigma}_{prin}(\boldsymbol{s}))}\right)^2\right) \tag{11}$$

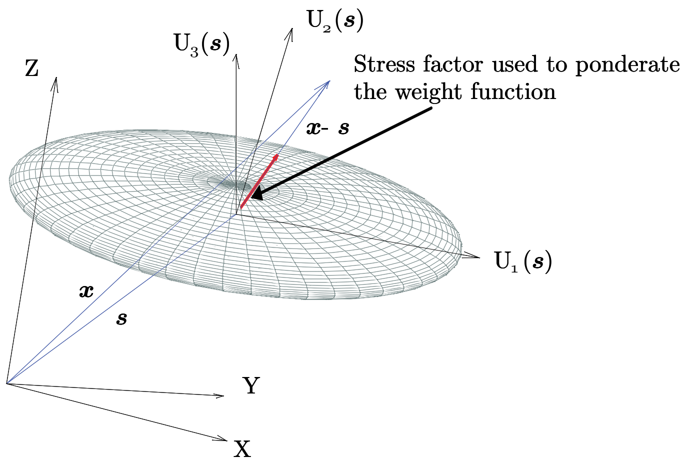


Figure 1: Definition of the ρ coefficient giving the influence of $\boldsymbol{s}$ on $\boldsymbol{x}$

with $\rho(\boldsymbol{x}, \boldsymbol{\sigma}_{prin}(\boldsymbol{s}))$ equal to the radial coordinate of the ellipsoid defined previously in the direction $(\boldsymbol{x} - \boldsymbol{s})$.
The intensity of the influence of a point at $\boldsymbol{s}$ on its neighborhood depends on the magnitude and direction of the principal stresses at $\boldsymbol{s}$.

3 INITIATION OF FAILURE

In the framework of nonlocal elasticity, Eringen and coworkers [4] have pointed out that the point encountering the maximum stress is not located at the crack tip. Simone and coworkers [3] have extended this study to nonlocal damage models, showing that the bad description of the nonlocal field leads to a wrong initiation of damage. To illustrate this problem and to compare the numerical solution of the original and the stress based nonlocal method, a notched plate under tension is studied (Fig. 2) with a pre-existing crack of length $h = 0.0005$ m. Due to symmetry, only half of the specimen is described.

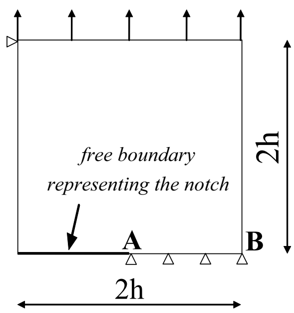


Figure 2: Compact Tension Specimen (CTS)

The notch is described geometrically by letting free the boundary. The influence of the

internal length of the model on the location of the maximum nonlocal equivalent strain is studied. The following parameters are used for the material: E = 1000 MPa; $\nu = 0.2$; $l_c = 0.0001$, 0.0002 or 0.0005 m.
The equivalent strain defined by Mazars is calculated from the strain field obtained under an imposed displacement. The nonlocal equivalent strain is then computed according to Eq. 3. The evolution along the line AB in front of the crack is given on Fig 3 for both nonlocal methods. These results, obtained by Simone et al., show a shift of the

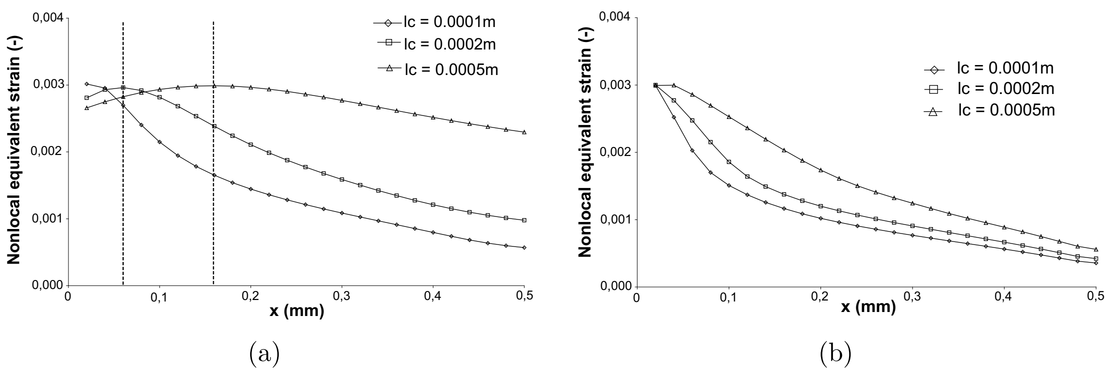


Figure 3: CTS: Evolution of the nonlocal equivalent strain along AB. (a) Original nonlocal method; (b) Stress based nonlocal method.

maximum nonlocal equivalent strain with the original nonlocal method leading thus to a wrong location of damage initiation. Furthermore, this shift is proportional to the internal length.
In the original nonlocal method, the domain of interactions depends only on the distance between points. So, a point at the crack tip will be influenced in the same way by points in the shadow zone of the notch than by points in front of the notch. Since the strain gradient is smaller in front of the notch than at the back, the maximum nonlocal equivalent strain is shifted to the notch front.
For the same test, with the stress based nonlocal method, the shift is null whatever the characteristical length l_c chosen. Indeed, the points in the shadow zone of the notch no more influence the point at the crack tip since they encounter a low stress state.
This study shows the capability of the stress based nonlocal method to correctly locate the inception of material nonlinearities in mode I problem with a pre-existing crack. This result is a key issue for size effect analysis.

4 SIZE EFFECT ANALYSIS: 3 POINT BENDING TEST

Size effects have been widely studied in the past regarding concrete structure and it is a key issue when one wants to transcript the material behavior identified at the scale of a laboratory specimen to a real structure.
The use of nonlocal models allows to reproduce this size effect by introducing an internal

length. However, we can observe that the parameters obtained depend a lot on the geometry and the stress state. Indeed, it has been shown the original nonlocal model show discrepancies when we compare results from one type of test to another (e.g. notched and unnotched beam) [5]. A first attempt to improve the results regarding size effect has been made by Krayani and coworkers [5]. By modifying the area of regularization close to free boundaries, a better redistribution of the state parameter was obtained leading to an improvement of the results.
We compare the original and the stress based nonlocal damage formulation through 3 point bending test on unnotched and notched beams with three geometrically similar sizes. The specimens with constant depth (b = 1 m), various heights (D = 80, 160, 320 mm) and corresponding lengths (L = 3D) are referred to as small, medium and large beam, respectively. Simulations are performed in 2D plane stress conditions. The model parameters used for these simulations are the same as the one in [5] : $E = 3.85 \times 10^4 MPa$, $\nu = 0.24$, $A_t = 0.95$, $A_c = 1.25$, $B_t = 9200$, $B_c = 1000$, $\varepsilon_{D_0} = 3.0 \times 10^{-5}$ and $l_c = 10$ mm.

4.1 Size effect on unnotched specimens

From the peak load P_u, we estimate the nominal strength σ_N, according to the elastic beam theory (Tab. 1):

$$\sigma_N = \frac{9}{2}\frac{P_u}{bD} \tag{12}$$

We notice that the results are quite close to each other, since the main difference is

Table 1: Unnotched beam: Peak load and nominal strength for the two nonlocal methods

	Original nonlocal		Stress based nonlocal	
D(mm)	P_u(kN)	σ_N(MPa)	P_u(kN)	σ_N(MPa)
80	65.732	3.70	64.380	3.62
160	123.792	3.48	122.240	3.43
320	240.440	3.38	218.360	3.07

only for the interactions perpendicular to the lower free boundary. With the stress based nonlocal model, the interactions are along lines parallel to the bottom side of the beam thus to a smaller domain of interactions and a lower peak load.
We use now the Bazant's size effect law for the case of unnotched beams [8]:

$$\sigma_N = f_{r\infty}\left(1 + \frac{D_b}{D}\right) \tag{13}$$

where D_b and $f_{r\infty}$ are constants. These two constants are obtained from a linear regression.

Table 2: Unnotched beam: Identification of D_b and $f_{r\infty}$ for the two nonlocal methods

	Original nonlocal	Stress based nonlocal
$D_b(mm)$	10.34	18.31
$f_{r\infty}(MPa)$	3.30	2.98

4.2 Size effect on notched specimens

The size effect analysis is now performed on notched specimen with a $0.2D$ high notch located at mid-span. From the peak load P_u, we estimate the nominal strength σ_N, according to the elastic beam theory (Tab. 3):

$$\sigma_N = \frac{9}{2}\frac{P_u}{bD(0.8^2)} \tag{14}$$

Again using the Bazant's size effect law for notched beams [8]:

Table 3: Notched beam: Peak load and nominal strength for the two nonlocal methods

	Original nonlocal		Stress based nonlocal	
D (mm)	P_u (kN)	σ_N (MPa)	P_u (kN)	σ_N (MPa)
80	42.316	3.72	35.630	3.13
160	64.426	2.83	59.130	2.60
320	97.454	2.14	94.440	2.07

$$\sigma_N = \frac{Bf_{r\infty}}{\sqrt{\left(1 + \frac{D}{D_0}\right)}} \tag{15}$$

where B is a dimensionless geometry-dependent parameter and D_0 is a characteristic size. For each formulation, D_0 and $Bf_{r\infty}$ are obtained from a linear regression. B can

Table 4: Notched beam: Identification of D_0 and $Bf_{r\infty}$ for the two nonlocal methods

	Original nonlocal	Stress based nonlocal
$D_0(mm)$	42.42	110.87
$Bf_{r\infty}(MPa)$	6.25	4.08

be calculated according to Rilem recommendations ($B = 1.08$) [9]. One can compare the values of $Bf_{r\infty}$ obtained with unnotched and notched specimen and give an error committed by the model regarding the description of size effect. Indeed, $f_{r\infty}$ is a parameter

Table 5: Comparison of the values $Bf_{r\infty}$ obtained from unnotched and notched specimens for the two nonlocal method

$Bf_{r\infty}$(MPa)	Original nonlocal	Stress based nonlocal
Computed from unnotched	3.56	3.21
Fit from notched	6.25	4.08
Relative error	76%	27%

relative to the material and should not be affected by size effect. Another mean to observe the consistency of the formulations regarding size effect is to use the Bazant's universal size effect law [8]:

$$\sigma_N = \frac{Bf_{r\infty}}{\sqrt{\left(1+\frac{D}{D_0}\right)}}.\left(1+\left(\left(\eta+\frac{D}{D_b}\right).\left(1+\frac{D}{D_0}\right)\right)^{-1}\right) \tag{16}$$

We consider in this formula the parameters linked to the unnotched specimen as known with the first analysis and we fit the value of D_0 in order to retrieve the values of the nominal strength of the notched specimens. The best fits are obtained for $D_0 = 300$ mm

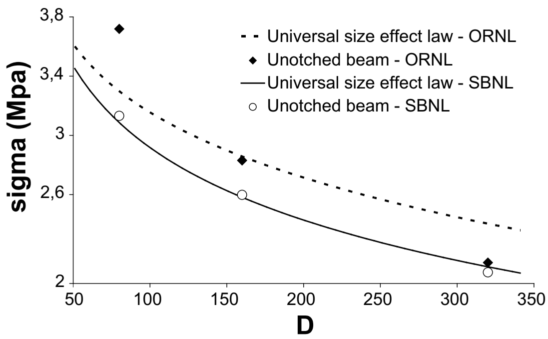
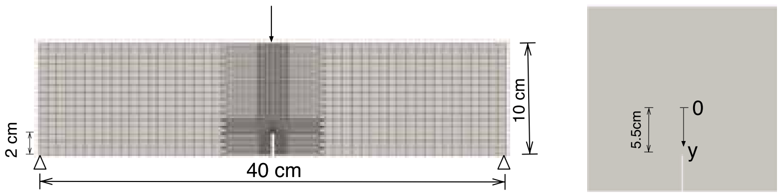


Figure 4: Fits of the universal size effect law on notched specimens for the original nonlocal method (ORNL) and the stress based nonlocal method(SBNL)

with the original nonlocal formulation and $D_0 = 255$ mm with the stress based nonlocal formulation. A better fit of the universel size effect law on the notched beam results is obtained with the stress based nonlocal model. It confirms the results obtained with the first method of error estimator in the description of size effect.
This analysis shows the internal length, that is linked to the size of the FPZ and allows to describe size effect, depends on the material through a characteristical length l_c as it is defined in the original version but also on the stress state as it is introduced in the stress based nonlocal model.

5 CRACKING EVOLUTION IN A 3 POINT BENDING TEST OF A NOTCHED BEAM

The efficiency of the stress based nonlocal method to describe the failure process of quasi-brittle material is explored through the comparison with experimental results on the evolution of crack opening along the height of a notched beam under three point bending test (3PBT). This beam tested at the GEM laboratory in Nantes by the second author is depicted in Fig. 5(a) with the associated mesh used for the numerical investigation. This beam has been modeled in 2D under plane stress conditions. The loading has been



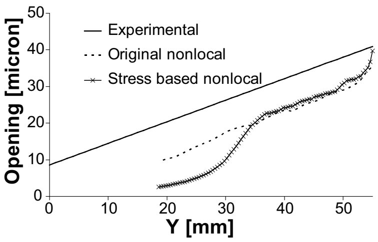
(a) (b)

Figure 5: Three point bending test: (a) Mesh of the specimen. (b) Axis of the crack opening.

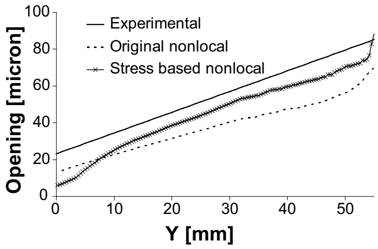
applied via displacement control. Both nonlocal methods have been successively used with the same set of parameters for the damage model describing concrete (see Eq. 5): E = 30,000 MPa; ν = 0.24; ε_{D_0} = 0.00004; β = 1.06; A_t = 0.9; B_t = 4000; A_c = 1.25; B_c = 1000 and l_c = 0.008 m.

The beam is composed of isoparametric elements with linear interpolation. A peculiar attention was taken to describe finely the notch tip where stress concentration occurs.

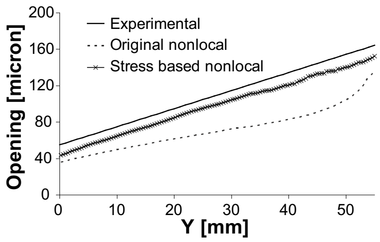
Image correlation technique was used during the experiment to follow the strain localisation process. Fig. 5(b) gives the orientation and the origin along the height of the beam of the measured cracking. We compare hereafter these experimental results with the one obtained from numerical calculations using the post treatment method proposed by Dufour and coworkers [10]. Fig. 6b, c and d show the crack opening along the height at different CMOD levels. One can observe that for the lowest CMOD considered, we have a cracking prediction close between the original and stress based nonlocal method. We are still in a diffused microcracking far from the crack tip that leads to a strain field similar between both methods. During the localisation process, the stress level and the internal length of the stress based nonlocal decreases, as a consequence we obtained a sharper strain profile that is closer to the one of a macrocrack for higher CMOD [11]. It leads so to a better estimation of the crack opening. A reason for the constant shift (error around 10%) between the experimental and the numerical results regarding the crack opening can be due to the 2D approximation used for the calculation. Indeed, experimentally, the crack opening was measured on surface and we can expect to have different values in the depth.



(a)



(b)



(c)

Figure 6: 3PBT: Crack opening along the height of the beam at different loading level. (a) 50 μm; (b) 100 μm and (c) 200 μm

6 CONCLUSIONS

We have proposed in this paper a modification of the integral nonlocal model in order to adapt the regularisation close to free boundary and during the cracking process. The stress state of each point is used during the calculation in order to create an evolution of the interaction between points. Each point interacts with its neighborhood in function of the intensity and direction of its principal stress values. All these improvments are made with no additional parameter that would be difficult to calibrate and the computational cost is similar to the original nonlocal method since the connectivity maxtric is identical. This can be easily implemented in any FE code that already includes nonlocal approach. The modification has been illustrated through several examples. The stress based nonlocal approach provides a better solution to model damage initiation. The proposed approach is capable to perfectly locate the damage initiation which is badly estimated with any other regularisation techniques. This result is a key issue when considering size effect analysis. The proposed approach gives a better description of size effect compared to the original one.

Furthermore, damage and strain profiles across a fracture process zone are more physically sounded. The result objectivity is conserved and our proposal allows to obtain a weak discontinuity that gets close to strong discontinuity upon mesh refinement. As a result the estimate of crack opening is much improved. This is a key issue as information on crack opening is a hot topic in structural engineering for durability analysis or the leakage rate estimation for confining vessels. It has been shown that continuous modelling can provide this kind of information. By improving the FE calculation and more particularly by taking into account the effect of a damaged zone on its vicinity, crack opening estimation are greatly improved.

In future works, loaded interface between two materials and loaded boundaries will be investigated. This will be of great interest for structural analysis with interaction between a crack in concrete and rebars.

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